Journal of Combinatorial Theory, Series B 99 (2009) 324–337

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Judicious k-partitions of graphs

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ARTICLE INFO

Article history:
Received 24 August 2007

Keywords:
Graph
Graph partition
Judicious partition

ABSTRACT

Judicious partition problems ask for partitions of the vertex set of graphs so that several quantities are optimized simultaneously. In this paper, we answer the following judicious partition question of Bollobás and Scott [B. Bollobás, A.D. Scott, Problems and results on judicious partitions, Random Structures Algorithms 21 (2002) 414–430] in the affirmative: For any positive integer k and for any graph G of size m, does there exist a partition of $V(G)$ into $V_1, \dots, V_k$ such that the total number of edges joining different V_i is at least $\frac{k-1}{k}m$, and for each $i \in \{1, 2, \dots, k\}$ the total number of edges with both ends in V_i is at most

$$\frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)?$$

We also point out a connection between our result and another judicious partition problem of Bollobás and Scott [B. Bollobás, A.D. Scott, Problems and results on judicious partitions, Random Structures Algorithms 21 (2002) 414–430].

1. Introduction

Bollobás and Scott [4] (also see [12]) introduced and studied *judicious* partition problems: Given a graph G, find a partition of $V(G)$ into $V_1, \dots, V_k$ such that for some $1 \leqslant t \leqslant k$ all collections $\{V_{i_1}, \dots, V_{i_t}\}$ satisfy certain constraints. For example, the *Maximum Bipartite Subgraph Problem* can be formulated as the following judicious partition problem: Given a graph G, find a partition of $V(G)$

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[1] Supported by Nanjing Normal University, NSFC, and Georgia Institute of Technology.
[2] Partially supported by NSA and by NSFC Project 10628102.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2008.08.007

into V_1, V_2 that minimizes $e(V_1) + e(V_2)$, where, for each $i \in \{1, 2\}$, $e(V_i)$ denotes the number of edges of G with both ends in V_i.

It is a well known fact that every graph with m edges contains a bipartite subgraph with at least $m/2$ edges. Edwards [8,9] improved this lower bound to $m/2 + h(m)/4$; here and throughout this paper

$$h(m) = \sqrt{2m + \frac{1}{4}} - \frac{1}{2}.$$

This bound is best possible for infinite many values of m as evidenced by the complete graphs K_{2n+1}. On the other hand, Alon [1] showed that the gap between this bound and the truth could be arbitrarily large, confirming a conjecture of Erdös. Maximum bipartite subgraphs in weighted graphs have also been studied, see for example [3,6].

In [7] (also see [6]), Bollobás and Scott extend Edwards' bound to k-partitions of graphs: The vertex set of any graph with m edges can be partitioned into $V_1, \dots, V_k$ such that

$$e(V_1, \dots, V_k) := \sum_{1 \leqslant i < j \leqslant k} e(V_i, V_j) \geqslant \frac{k-1}{k} m + \frac{k-1}{2k} \sqrt{2m + \frac{1}{4}} - \frac{k^2 - 2k + 2}{8k}, \tag{1.1}$$

where $e(V_i, V_j)$ is the number of edges with one end in V_i and the other in V_j. Clearly, $e(V_1, \dots, V_k)$ is the number of edges of G that join vertices from different V_i. We point out that in Theorem 24 of [7], the term $-\frac{k^2-2k+2}{8k}$ in (1.1) is printed as $+\frac{k^2-2k+2}{8}$, which was the result of a typo in the final line of the calculation of $f_k(K_n)$ (the equation above Theorem 24 in [7]). The bound in (1.1) presented here is the correct one.

In [5], Bollobás and Scott consider the following judicious partition problem: Given a graph G, find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$. This is the *Bottleneck Bipartition Problem* asked by Entringer (see [13]). Porter [10] proved that for any graph G with m edges there is a partition V_1, V_2 of $V(G)$ such that $\max\{e(V_1), e(V_2)\} \leqslant m/4 + O(\sqrt{m})$, establishing a conjecture of Erdös. Shahrokhi and Székely [13] proved that this problem is NP-hard. Alon et al. [2] proved that graphs with large bipartite subgraphs also have good judicious partitions. More precisely, if a graph of size m has a bipartite subgraph with $m/2 + \delta$ edges, then its vertex set can be partitioned into V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leqslant m/4 - (1 - o(1))\delta/2 + O(\sqrt{m})$ (when $\delta = o(m)$) and $\max\{e(V_1), e(V_2)\} \leqslant (1/4 - \Omega(1))m$ (when $\delta = \Omega(m)$).

Bollobás and Scott [5] proved the following result, which says that one can always find a bipartition of any graph which satisfies both the Edwards bound and a best possible upper bound on $\max\{e(V_1), e(V_2)\}$. We use $N(x)$ to denote the neighborhood of the vertex x in a graph.

Theorem 1.1. *(See Bollobás and Scott [5].) Let G be a graph with m edges. Then there is partition V_1, V_2 of $V(G)$ such that*

(1) $|N(x) \cap V_2| \geqslant |N(x) \cap V_1|$ *for all* $x \in V_1$,
(2) $e(V_i) \leqslant \frac{m}{4} + \frac{1}{8} h(m)$ *for* $i = 1, 2$, *and*
(3) $e(V_1, V_2) \geqslant \frac{m}{2} + \frac{1}{4} h(m)$.

Condition (1) is essential in the proof of Theorem 1.1 in [5]. Moreover, the bounds in (2) and (3) are (individually) tight; and the complete graphs K_{2n+1} are the only extremal graphs (modulo isolated vertices) for Theorem 1.1.

For general k-partitions, Bollobás and Scott [5] also proved that for any integer $k \geqslant 1$ and any graph G of size m, $V(G)$ can be partitioned into $V_1, \dots, V_k$ such that for each $i \in \{1, 2, \dots, k\}$,

$$e(V_i) \leqslant \frac{m}{k^2} + \frac{k-1}{2k^2} h(m). \tag{1.2}$$

Again, the complete graphs of order $kn + 1$ are the only extremal graphs (modulo isolated vertices).

Porter [11] showed that if k is a power of 2 then every graph G with m edges has a partition of $V(G)$ into $V_1, \dots, V_k$ such that

$$e(V_1,\ldots,V_k) \geqslant \frac{k-1}{k}m$$

and for $i \in \{1,\ldots,k\}$,

$$e(V_i) \leqslant \frac{m}{k^2} + \sqrt{m/k}.$$

As noted in [5], Theorem 1.1 can be used to show that this result of Porter's remains valid when the bound on $e(V_i)$ is replaced by (1.2).

For the general case, Bollobás and Scott [6] asked the following.

Problem 1.2. (See Bollobás and Scott [6].) Does any graph G of size m have a partition of $V(G)$ into $V_1,\ldots,V_k$ that satisfy both (1.1) and (1.2)?

As a possibly easier question, Bollobás and Scott [6] also asked the following.

Problem 1.3. (See Bollobás and Scott [6].) Does any graph G of size m have a partition of $V(G)$ into $V_1,\ldots,V_k$ such that (1.2) holds and

$$e(V_1,\ldots,V_k) \geqslant \frac{k-1}{k}m?$$

As noted above, a result in [5] shows that the answer to Problem 1.3 is "Yes" when k is a power of 2. The following theorem is the main result of this paper, which gives an affirmative answer to Problem 1.3.

Theorem 1.4. *Let G be a graph of size m, and let $k \geqslant 1$ be an integer. Then $V(G)$ can be partitioned into $V_1,\ldots,V_k$ such that*

(1) *for each $i \in \{1,\ldots k-1\}$ and for every $x \in V_i$, $|N(x) \cap (\bigcup_{j=i+1}^{k} V_j)| \geqslant (k-i)|N(x) \cap V_i|$,*
(2) *for each $i \in \{1,2,\ldots,k\}$, $e(V_i) \leqslant \frac{1}{k^2}m + \frac{k-1}{2k^2}h(m)$, and*
(3) *$e(V_1,\ldots,V_k) \geqslant \frac{k-1}{k}m + \frac{1}{2k}h(m)$.*

Note that when $k=2$, Theorem 1.4 becomes Theorem 1.1. We will need the following result, which is a consequence of (1.1) and (1.2).

Lemma 1.5. *Let $k \geqslant 2$ and $m \geqslant 0$ be integers. Then for any graph G with m edges there is a partition $V_1,\ldots,V_k$ of $V(G)$ such that*

$$e(V_1,\ldots,V_k) \geqslant \frac{k-1}{k}m + \frac{1}{2k}h(m).$$

Proof. First, assume $m \geqslant (k+2)^2/8$. Then

$$\begin{aligned}\frac{k-2}{2k}\sqrt{2m+1/4} - \frac{k-2}{8} &\geqslant \frac{k-2}{4k}\sqrt{(k+2)^2+1} - \frac{k-2}{8}\\ &> \frac{k^2-4}{4k} - \frac{k-2}{8}\\ &= \frac{k^2+2k-8}{8k}\\ &\geqslant 0 \quad (\text{since } k \geqslant 2).\end{aligned}$$

So by (1.1),

$$\begin{aligned}
e(V_1,\ldots,V_k) &\geqslant \frac{k-1}{k}m+\frac{k-1}{2k}\sqrt{2m+1/4}-\frac{k^2-2k+2}{8k}\\
&=\frac{k-1}{k}m+\frac{1}{2k}h(m)+\frac{k-2}{2k}\sqrt{2m+1/4}+\frac{1}{4k}-\frac{k^2-2k+2}{8k}\\
&=\frac{k-1}{k}m+\frac{1}{2k}h(m)+\frac{k-2}{2k}\sqrt{2m+1/4}-\frac{k-2}{8}\\
&>\frac{k-1}{k}m+\frac{1}{2k}h(m).
\end{aligned}$$

Now assume $m<(k+2)^2/8$. Then by (1.2), there is a partition $V_1,\ldots,V_k$ of $V(G)$ such that for each $1\leqslant i\leqslant k$,

$$\begin{aligned}
e(V_i) &\leqslant \frac{1}{k^2}m+\frac{k-1}{2k^2}h(m)\\
&<\frac{(k+2)^2}{8k^2}+\frac{k-1}{2k^2}\left(\sqrt{\frac{(k+2)^2}{4}+\frac{1}{4}}-\frac{1}{2}\right)\\
&<\frac{(k+2)^2}{8k^2}+\frac{k-1}{4k^2}\big((k+3)-1\big)\\
&=\frac{3}{8}+\frac{3}{4k}<1 \quad (\text{since } k\geqslant 2).
\end{aligned}$$

Hence, $e(V_i)=0$ for all $i=1,\ldots,k$. Therefore,

$$e(V_1,\ldots,V_k)=m\geqslant\frac{k-1}{k}m+\frac{1}{2k}h(m),$$

completing the proof. □

We prove Theorem 1.4 in Section 3. In Section 2, we prove four inequalities to be used in the proof of Theorem 1.4. In Section 4, we discuss some interesting consequences and related problems.

For convenience, any partition $V_1,\ldots,V_k$ of $V(G)$ satisfying (1) above is said to satisfy the *property* P(k). Note that such a partition always exists; for example, take a partition $V_1,\ldots,V_k$ of $V(G)$ such that $e(V_1,\ldots,V_k)$ is maximum.

2. Four inequalities

In this section we prove four elementary (but nontrivial) inequalities. These inequalities will be used in the next section to prove Theorem 1.4.

Lemma 2.1. *Let $k\neq 0, m, q$ be real numbers such that $h(m)$ is also a real number, and let $m'=\frac{(k-1)^2}{k^2}m+q$. If*

$$q\leqslant\frac{k-1}{2k^2}h(m)$$

then

$$\frac{1}{(k-1)^2}m'+\frac{k-2}{2(k-1)^2}h(m')\leqslant\frac{m}{k^2}+\frac{k-1}{2k^2}h(m).$$

Proof. Note that

$$\begin{aligned}
\frac{m'}{(k-1)^2}+\frac{k-2}{2(k-1)^2}h(m') &=\frac{m}{k^2}+\frac{q}{(k-1)^2}+\frac{k-2}{2(k-1)^2}\left(\sqrt{\frac{2(k-1)^2}{k^2}m+2q+\frac{1}{4}}-\frac{1}{2}\right)\\
&=\frac{m}{k^2}+\frac{k-1}{2k^2}h(m)-f(q),
\end{aligned}$$

where

$$f(q)=\frac{k-1}{2k^2}h(m)-\frac{q}{(k-1)^2}-\frac{k-2}{2(k-1)^2}\left(\sqrt{\frac{2(k-1)^2}{k^2}m+2q+\frac{1}{4}}-\frac{1}{2}\right).$$

We further write $g(q)=2k^2(k-1)^2f(q)$. If $g(q)\geqslant 0$ then $f(q)\geqslant 0$; and hence the assertion of the lemma holds. So it suffices to show that if $g(q)<0$ then $q>\frac{k-1}{2k^2}h(m)$.

Therefore, we may assume $g(q)<0$. Then a simple calculation shows that

$$(k-1)^3\sqrt{2m+\frac{1}{4}}-2k^2q+\frac{k^2-3k+1}{2}<k(k-1)(k-2)\sqrt{2m+\frac{2k^2q}{(k-1)^2}+\frac{k^2}{4(k-1)^2}}.$$

By squaring both sides and combining like terms, we can express this inequality as the quadratic inequality $aq^2+bq+c<0$, where

$$a=4k^4,$$
$$b=-2k^2(k-1)\left(2(k-1)^2\sqrt{2m+\frac{1}{4}}+k^3-3k^2+2k-1\right),$$

and

$$c=2(k-1)^6m-2k^2(k-1)^2(k-2)^2m+(k-1)^3\left(k^2-3k+1\right)\sqrt{2m+\frac{1}{4}}$$
$$+\frac{(k-1)^6+(k^2-3k+1)^2-k^4(k-2)^2}{4}.$$

With straightforward calculations, we can show that

$$b^2-4ac=\left(2k^3(k-1)(k-2)\left(2\sqrt{2m+\frac{1}{4}}+k-1\right)\right)^2.$$

Therefore, a simple calculation gives

$$q>\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{k-1}{2k^2}h(m).\qquad\square$$

Lemma 2.2. *Let $k\neq 0$, δ, h be real numbers. Suppose*

$$2k^2\delta^2-k^2(2h+k-2)\delta+2(k-1)h^2+k(k-2)h<0.$$

Then

$$\frac{h}{k}<\delta<\frac{4kh+2k^2-4h-4k}{4k}.$$

Proof. Let $a=2k^2$, $b=-k^2(2h+k-2)$, and $c=2(k-1)h^2+k(k-2)h$. So $a\delta^2+b\delta+c<0$. Simple calculations show that

$$b^2-4ac=\left(2hk^2+k^3-2k^2-4kh\right)^2.$$

Therefore, by applying the quadratic formula, we have

$$\frac{k^2(2h+k-2)-(2hk^2+k^3-2k^2-4kh)}{4k^2}<\delta<\frac{k^2(2h+k-2)+(2hk^2+k^3-2k^2-4kh)}{4k^2}.$$

From this, the assertion of the lemma follows. $\square$

Lemma 2.3. *Let $m \geqslant 3$ and $k \geqslant 1$ be real numbers, and let*

$$g(m) = \frac{\sqrt{8(2k-1)^2 m + k^2} - k}{\sqrt{2m+1/4} - 1/2}.$$

Then

$$\frac{k-1}{(2k-1)^2} g(m) < \frac{5}{4}.$$

Proof. By differentiating $g(m)$, we have

$$g'(m) = \frac{f(m)}{\sqrt{8(2k-1)^2 m + k^2}\sqrt{2m+\frac{1}{4}}(\sqrt{2m+\frac{1}{4}} - \frac{1}{2})^2},$$

where

$$\begin{aligned}
f(m) &= (2k-1)^2\left(8m + 1 - 2\sqrt{2m+\frac{1}{4}}\right) - \left(8(2k-1)^2 m + k^2 - k\sqrt{8(2k-1)^2 m + k^2}\right) \\
&= (2k-1)^2 - 2(2k-1)^2\sqrt{2m+\frac{1}{4}} - k^2 + 2k(2k-1)\sqrt{2m + \frac{k^2}{4(2k-1)^2}} \\
&< (k-1)(3k-1) - 2(2k-1)^2\sqrt{2m+\frac{1}{4}} + 2k(2k-1)\sqrt{2m+\frac{1}{4}} \quad (\text{since } k \geqslant 1) \\
&= (k-1)\left(3k - 1 - (4k-2)\sqrt{2m+\frac{1}{4}}\right) \\
&< 0 \quad (\text{since } m \geqslant 3 \text{ and } k \geqslant 1).
\end{aligned}$$

Therefore, $g'(m) < 0$. Hence

$$\begin{aligned}
\frac{k-1}{(2k-1)^2} g(m) &\leqslant \frac{k-1}{(2k-1)^2} g(3) \quad (\text{since } m \geqslant 3) \\
&= \frac{k-1}{(2k-1)^2} \cdot \frac{\sqrt{24(2k-1)^2 + k^2} - k}{\sqrt{6+1/4} - 1/2} \\
&< \frac{9k(k-1)}{2(2k-1)^2} \\
&< \frac{5}{4}. \qquad \square
\end{aligned}$$

Lemma 2.4. *Let $k \neq 0$, m, α be real numbers such that $h(m)$ is also a real number, and let $\delta = \sqrt{2m/k^2 + 2\alpha + 1/4} - 1/2$. If*

$$\alpha \geqslant \frac{k-1}{2k^2} h(m),$$

then

$$\frac{2k-1}{2}\delta^2 + \frac{1}{2}\delta + \frac{k-1}{2k^2} h(m) \geqslant \frac{2k-1}{k^2} m.$$

Proof. Note that

$$\frac{2k-1}{2}\delta^2 + \frac{\delta}{2} + \frac{k-1}{2k^2} h(m) = \frac{2k-1}{k^2} m + f(k),$$

where

$$f(k)=(2k-1)\alpha+\frac{k-1}{2k^2}h(m)+\frac{k-1}{2}-(k-1)\sqrt{\frac{2m}{k^2}+2\alpha+\frac{1}{4}}.$$

Therefore, it suffices to show that if $f(k)<0$ then $\alpha<\frac{k-1}{2k^2}h(m)$. So let $f(k)<0$. Then

$$(2k-1)\alpha+\frac{k-1}{2k^2}\sqrt{2m+\frac{1}{4}}+\frac{(k-1)(2k^2-1)}{4k^2}<(k-1)\sqrt{\frac{2m}{k^2}+2\alpha+\frac{1}{4}}.$$

By squaring both sides and simplifying, we may write this inequality in the form $a\alpha^2+b\alpha+c<0$, where

$$a=4k^4(2k-1)^2,$$
$$b=2k^2(k-1)\left(2(2k-1)\sqrt{2m+\frac{1}{4}}+2k^2-2k+1\right),$$

and

$$c=(k-1)^2\left(2k^2-1\right)\sqrt{2m+\frac{1}{4}}-2(k-1)^2\left(4k^2-1\right)m-\frac{(k-1)^2(2k^2-1)}{2}.$$

With straightforward calculations we can show that

$$b^2-4ac=\left(4k^3(k-1)\left(2(2k-1)\sqrt{2m+\frac{1}{4}}-(k-1)\right)\right)^2.$$

Then a simple calculation shows that

$$\alpha<\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{k-1}{2k^2}h(m). \qquad \square$$

3. Partitions

In this section, we prove Theorem 1.4. The idea of our proof is to work with a partition $V_1,\dots,V_k$ of $V(G)$ for which the property P(k) holds and $e(V_1)\geqslant e(V_i)$ for all $i\in\{1,\dots,k\}$. Such a partition may be produced by maximizing $e(V_1,\dots,V_k)$. If $e(V_1)\leqslant\frac{1}{k^2}m+\frac{k-1}{2k^2}h(m)$ then (by Lemma 1.5) we have the desired partition. Otherwise, we move a vertex from V_1 to $\overline{V_1}:=V(G)-V_1$, and (inductively) partition $G[\overline{V_1}]$ (the subgraph of G induced by $\overline{V_1}$) into $k-1$ sets satisfying the property P($k-1$).

First, we prove a lemma about partitions that satisfy the property P(k) (defined at the end of Section 1). For a set $S\subseteq V(G)$ and $v\in V(G)$, we write $S-v:=S\setminus\{v\}$ and $S+v:=S\cup\{v\}$.

Lemma 3.1. *Let G be a graph, and let $V_1,\dots,V_k$ be a partition of $V(G)$ satisfying the property P(k). Then*

(1) $e(V_1,\overline{V_1})\geqslant 2(k-1)e(V_1)$, *and*
(2) *for any $v\in V_1$, $e(V_1-v,\overline{V_1}+v)\geqslant 2(k-1)e(V_1)-(k-2)|N(v)\cap V_1|$.*

Proof. Applying the property P(k) with $i=1$, we have

$$\left|N(x)\cap\overline{V_1}\right|\geqslant(k-1)\left|N(x)\cap V_1\right|$$

for all $x\in V_1$. Hence

$$e(V_1,\overline{V_1})=\sum_{x\in V_1}\left|N(x)\cap\overline{V_1}\right|\geqslant(k-1)\sum_{x\in V_1}\left|N(x)\cap V_1\right|=2(k-1)e(V_1),$$

and (1) holds. Moreover,

$$\begin{aligned} e(V_1 - v, \overline{V_1} + v) &= \left(\sum_{x \in V_1 - v} \left|N(x) \cap \overline{V_1}\right| \right) + \left|N(v) \cap V_1\right| \\ &\geqslant \left(\sum_{x \in V_1 - v} (k-1)\left|N(x) \cap V_1\right| \right) + \left|N(v) \cap V_1\right| \\ &= 2(k-1)e(V_1) - (k-2)\left|N(v) \cap V_1\right|. \qquad \square \end{aligned}$$

We now prove Theorem 1.4. We may assume $m \geqslant 3$; as Theorem 1.4 clearly holds when $m \leqslant 2$. We apply induction on k. Theorem 1.4 is certainly true when $k = 1$. When $k = 2$, Theorem 1.4 follows from Theorem 1.1. So we may assume $k \geqslant 3$ and that the assertion of Theorem 1.4 holds when partitioning any graph into $k - 1$ sets. That is, for any graph G',

(∗) $V(G')$ may be partitioned into $U_2, \ldots, U_k$ such that
- for $i \in \{2, \ldots, k-1\}$ and $x \in U_i$, $|N(x) \cap (\bigcup_{j=i+1}^{k} U_j)| \geqslant ((k-1) - (i-1))|N(x) \cap U_i|$,
- $e(U_i) \leqslant \frac{e(G')}{(k-1)^2} + \frac{k-2}{2(k-1)^2} h(e(G'))$ for all $i \in \{2, \ldots, k\}$, and
- $e(U_2, \ldots, U_k) \geqslant \frac{k-2}{k-1} e(G') + \frac{1}{2(k-1)} h(e(G'))$.

Let $V_1, \ldots, V_k$ be a partition of $V(G)$ which maximizes $e(V_1, \ldots, V_k)$. Without loss of generality, we may assume that

$$e(V_1) \geqslant \max_{2 \leqslant i \leqslant k} \{e(V_i)\}.$$

By the maximality of $e(V_1, \ldots, V_k)$, for any $1 \leqslant i \neq j \leqslant k$ and for any $x \in V_i$ we have $|N(x) \cap V_i| \geqslant |N(x) \cap V_j|$. So the partition $V_1, \ldots, V_k$ satisfies the property P(k), and hence (1) of Theorem 1.4 holds. By Lemma 1.5 (and the maximality of $e(V_1, \ldots, V_k)$), the partition $V_1, \ldots, V_k$ also satisfies (3) of Theorem 1.4.

Thus we may assume that

$$\begin{aligned} &V_1, \ldots, V_k \text{ is a partition of } V(G) \text{ satisfying (1) and (3) of Theorem 1.4,} \\ &e(V_1) \geqslant \max_{2 \leqslant i \leqslant k} \{e(V_i)\}, \\ &\text{and subject to this,} \quad e(V_1) \text{ is minimal.} \end{aligned} \tag{3.1}$$

Let $e(V_1) = \frac{m}{k^2} + \alpha$. If $\alpha \leqslant \frac{k-1}{2k^2} h(m)$, we are done. So we may assume that

$$\alpha > \frac{k-1}{2k^2} h(m). \tag{3.2}$$

Let H be the subgraph of G induced by V_1, and let v be a vertex of H with minimum nonzero degree δ. Let $W_1 := V_1 - v$. Then for every $x \in W_1$,

$$\left|N(x) \cap \overline{W_1}\right| \geqslant \left|N(x) \cap \overline{V_1}\right| \geqslant (k-1)\left|N(x) \cap V_1\right| \geqslant (k-1)\left|N(x) \cap W_1\right|. \tag{3.3}$$

Let $m' := e(\overline{W_1}) = m - e(W_1) - e(W_1, \overline{W_1})$. By Lemma 3.1(2),

$$e(W_1, \overline{W_1}) \geqslant 2(k-1)e(V_1) - (k-2)\delta. \tag{3.4}$$

Therefore, since $e(W_1) = e(V_1) - \delta = \frac{m}{k^2} + \alpha - \delta$,

$$m' \leqslant \frac{(k-1)^2}{k^2} m - (2k-1)\alpha + (k-1)\delta. \tag{3.5}$$

By (∗), $\overline{W_1}$ admits a partition $W_2, \ldots, W_k$ such that for every $i \in \{2, 3, \ldots, k-1\}$ and for every $x \in W_i$,

$$\left| N(x) \cap \left(\bigcup_{j=i+1}^{k} W_j \right) \right| \geqslant \big((k-1)-(i-1)\big)\big|N(x) \cap W_i\big| \tag{3.6}$$

(the property P($k-1$), where the neighborhood is taken in the subgraph of G induced by $\overline{W}_1$), and such that

$$e(W_i) \leqslant \frac{m'}{(k-1)^2} + \frac{k-2}{2(k-1)^2} h(m') \quad \text{for } i \in \{2, \ldots, k\}, \tag{3.7}$$

and

$$e(W_2, \ldots, W_k) \geqslant \frac{k-2}{k-1} m' + \frac{1}{2(k-1)} h(m'). \tag{3.8}$$

We wish to show that $W_1, \ldots, W_k$ give the desired partition of $V(G)$ for Theorem 1.4. By (3.3) and (3.6), we see that the partition $W_1, \ldots, W_k$ satisfies the property P(k); and so (1) of Theorem 1.4 holds for $W_1, \ldots, W_k$. Further, $W_1, \ldots, W_k$ satisfies (3) of Theorem 1.4:

$$\begin{aligned}
e(W_1, \ldots, W_k) &= e(W_1, \overline{W_1}) + e(W_2, \ldots, W_k) \\
&> e(W_1, \overline{W_1}) + \frac{k-2}{k-1} m' \quad \big(\text{by (3.8)}\big) \\
&= e(W_1, \overline{W_1}) + \frac{k-2}{k-1}\big(m - e(W_1) - e(W_1, \overline{W_1})\big) \\
&= \frac{k-2}{k-1} m - \frac{k-2}{k-1} e(W_1) + \frac{1}{k-1} e(W_1, \overline{W_1}) \\
&\geqslant \frac{k-2}{k-1} m - \frac{k-2}{k-1}\big(e(V_1) - \delta\big) + \frac{1}{k-1}\big(2(k-1)e(V_1) - (k-2)\delta\big) \quad \big(\text{by (3.4)}\big) \\
&= \frac{k-2}{k-1} m + \frac{k}{k-1} e(V_1) \quad \big(\text{since } e(V_1) = m/k^2 + \alpha\big) \\
&= \frac{k-1}{k} m + \frac{k}{k-1} \alpha \\
&> \frac{k-1}{k} m + \frac{1}{2k} h(m) \quad \big(\text{by (3.2)}\big).
\end{aligned}$$

If the partition $W_1, \ldots, W_k$ also satisfies (2) of Theorem 1.4, then $W_1, \ldots, W_k$ form the desired partition of $V(G)$ for Theorem 1.4. So we may assume that

$$\max_{1 \leqslant i \leqslant k} e(W_i) > \frac{m}{k^2} + \frac{k-1}{2k^2} h(m). \tag{3.9}$$

Suppose $\delta \leqslant \frac{2k-1}{k-1}\alpha$. Then $m' = e(\overline{W_1}) \leqslant \frac{(k-1)^2}{k^2} m$ (by (3.5)). So by Lemma 2.1,

$$\max_{2 \leqslant i \leqslant k} e(W_i) \leqslant \frac{m'}{(k-1)^2} + \frac{k-2}{2(k-1)^2} h(m') \leqslant \frac{m}{k^2} + \frac{k-1}{2k^2} h(m).$$

Hence by (3.9),

$$e(W_1) > \frac{m}{k^2} + \frac{k-1}{2k^2} h(m).$$

Since $\delta > 0$, $e(W_1) = e(V_1) - \delta < e(V_1)$; and hence $W_1, \ldots, W_k$ is a partition of $V(G)$ that contradicts the choice of $V_1, \ldots, V_k$ in (3.1).

Therefore, we must have

$$\delta > \frac{2k-1}{k-1}\alpha. \tag{3.10}$$

Then $e(W_1) = e(V_1) - \delta < \frac{m}{k^2}$; and hence by (3.9),

$$\max_{2\leqslant i\leqslant k} e(W_i) > \frac{m}{k^2} + \frac{k-1}{2k^2}h(m). \tag{3.11}$$

Note from (3.5) and (3.7) that $\max_{2\leqslant i\leqslant k} e(W_i) \leqslant \frac{m}{k^2} + f(\alpha)$, where

$$f(\alpha) = -\frac{2k-1}{(k-1)^2}\alpha + \frac{\delta}{k-1} + \frac{k-2}{2(k-1)^2}\left(\sqrt{\frac{2(k-1)^2}{k^2}m - (4k-2)\alpha + (2k-2)\delta + \frac{1}{4}} - \frac{1}{2}\right).$$

So by (3.11), $f(\alpha) > \frac{k-1}{2k^2}h(m)$. For convenience, we simply write h for $h(m)$. Since $f'(\alpha) < 0$ and $\alpha > \frac{k-1}{2k^2}h$ (by (3.2)), $f(\frac{k-1}{2k^2}h) > f(\alpha)$. So

$$f\left(\frac{k-1}{2k^2}h\right) > \frac{k-1}{2k^2}h.$$

By simplifying this inequality (and noting that $2m = h^2 + h$), we get

$$h - 2\delta + \frac{k-2}{2(k-1)} < \frac{k-2}{k}\sqrt{h^2 - \frac{kh}{k-1} + \frac{2k^2\delta}{k-1} + \frac{k^2}{4(k-1)^2}}.$$

Simplifying further, this inequality is reduced to $a\delta^2 + b\delta + c < 0$, where

$$a = 2k^2,$$
$$b = -k^2(2h + k - 2),$$

and

$$c = 2(k-1)h^2 + k(k-2)h.$$

By Lemma 2.2, we have

$$\frac{h}{k} < \delta < \frac{4kh + 2k^2 - 4h - 4k}{4k}. \tag{3.12}$$

Recall that H is the subgraph of G induced by V_1, and δ is the minimum nonzero degree in H. Let $V_1^* \subseteq V_1$ be the set of vertices with nonzero degree in H. Then $|V_1^*| \geqslant \delta + 1$.

We now prove that

$$\left|V_1^*\right| = \delta + 1. \tag{3.13}$$

Since $m/k^2 + \alpha = e(V_1) \geqslant \delta(\delta+1)/2$, we have

$$\delta \leqslant \sqrt{\frac{2m}{k^2} + 2\alpha + \frac{1}{4}} - \frac{1}{2}.$$

So by (3.10), $\sqrt{\frac{2m}{k^2} + 2\alpha + \frac{1}{4}} - \frac{1}{2} > \frac{2k-1}{k-1}\alpha$, which reduces to

$$k^2(2k-1)^2\alpha^2 + k^2(k-1)\alpha - 2(k-1)^2 m < 0.$$

Solving this inequality, we get

$$\alpha < \frac{(k-1)\sqrt{8(2k-1)^2 m + k^2} - k(k-1)}{2k(2k-1)^2}. \tag{3.14}$$

Since $\delta > \frac{h}{k}$ (by (3.12)),

$$\frac{2m}{k^2} + 2\alpha = 2e(V_1) = \sum_{x \in V_1} d_H(x) > \frac{h}{k}\left|V_1^*\right|.$$

Hence

$$\begin{aligned}
\left|V_1^*\right| &< \frac{2m}{kh} + \frac{2k\alpha}{h} \\
&= \frac{2m + \frac{1}{4} - \frac{1}{4}}{kh} + \frac{2k\alpha}{h} \\
&< \frac{1}{k}\left(\sqrt{2m + \frac{1}{4}} + \frac{1}{2}\right) + \frac{2k\alpha}{h} \\
&= \frac{h}{k} + \frac{1}{k} + \frac{2k\alpha}{h} \\
&< \frac{h}{k} + \frac{1}{k} + \frac{k-1}{(2k-1)^2} g(m) \quad \bigl(\text{by (3.14)}\bigr),
\end{aligned}$$

where

$$g(m) = \frac{\sqrt{8(2k-1)^2 m + k^2} - k}{\sqrt{2m + 1/4} - 1/2}.$$

By Lemma 2.3,

$$\frac{k-1}{(2k-1)^2} g(m) < \frac{5}{4}.$$

Therefore,

$$\begin{aligned}
\left|V_1^*\right| &\leqslant \frac{h}{k} + \frac{1}{k} + \frac{5}{4} < \delta + \frac{1}{k} + \frac{5}{4} \quad \bigl(\text{by (3.12)}\bigr) \\
&< \delta + 2 \quad (\text{since } k \geqslant 3).
\end{aligned}$$

Since $|V_1^*|$ and δ are both integers, $|V_1^*| = \delta + 1$; and so we have (3.13).

By (3.13), H consists of the complete graph of order $\delta + 1$ and some isolated vertices. Moreover, every $x \in V_1^*$ has degree δ in H, and

$$\delta = \sqrt{\frac{2m}{k^2} + 2\alpha + \frac{1}{4}} - \frac{1}{2}. \tag{3.15}$$

Since $V_1, \ldots, V_k$ satisfies the property P(k), we have

$$\left|N(x) \cap \overline{V_1}\right| \geqslant (k-1)\left|N(x) \cap V_1\right| = (k-1)\delta$$

for every $x \in V_1^*$. So

$$\begin{aligned}
e(W_1, \overline{W_1}) &= \sum_{x \in W_1} \left|N(x) \cap \overline{W_1}\right| \\
&= \delta + \sum_{x \in V_1 - v} \left|N(x) \cap \overline{V_1}\right| \\
&\geqslant \delta + \sum_{x \in V_1^* - v} (k-1)\delta \\
&= \delta + (k-1)\delta^2.
\end{aligned}$$

Then

$$\begin{aligned}
m' &= m - e(W_1) - e(W_1, \overline{W_1}) \\
&\leqslant m - \frac{\delta^2 - \delta}{2} - (k-1)\delta^2 - \delta \\
&= \frac{(k-1)^2}{k^2} m + \frac{k-1}{2k^2} h(m) + \left(\frac{2k-1}{k^2} m - \frac{2k-1}{2}\delta^2 - \frac{1}{2}\delta - \frac{k-1}{2k^2} h(m) \right) \\
&\leqslant \frac{(k-1)^2}{k^2} m + \frac{k-1}{2k^2} h(m).
\end{aligned}$$

The last inequality follows from (3.15) and Lemma 2.4 (because $\alpha > \frac{k-1}{2k} h(m)$ by (3.2)). Therefore, by Lemma 2.1 and (3.7), we have

$$\max_{2 \leqslant i \leqslant k} e(W_i) \leqslant \frac{m}{k^2} + \frac{k-1}{2k^2} h(m).$$

This contradicts (3.11) and completes the proof. $\square$

4. Further results and related problems

Theorem 1.4 gives an affirmative answer to Problem 1.3. It would be interesting to know whether the lower bound on $e(V_1, \ldots, V_k)$ in Theorem 1.4 can be improved further. (In particular, it would be nice to know whether Problem 1.2 has a positive answer.) Our proof of Theorem 1.4 (at least parts of our proof) suggests that such an improvement may be possible, although we are not able to do so.

Theorem 1.4 also has the following consequence.

Corollary 4.1. *Let G be a graph with m edges, and let $k \geqslant 1$ be an integer. Then there is a partition of $V(G)$ into $V_1, \ldots, V_k$ such that for each $i \in \{1, \ldots, k\}$,*

$$\binom{k+1}{2} e(V_i) + \frac{k}{2} \sum_{j \neq i} e(V_j) \leqslant m + \frac{k-2}{4} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right).$$

Proof. By Theorem 1.4, let $V_1, \ldots, V_k$ be a partition of $V(G)$ such that

$$e(V_1, \ldots, V_k) \geqslant \frac{k-1}{k} m + \frac{1}{2k} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right),$$

and for each $i \in \{1, \ldots, k\}$,

$$e(V_i) \leqslant \frac{m}{k^2} + \frac{k-1}{2k^2} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right).$$

Then

$$\begin{aligned}
&\binom{k+1}{2} e(V_i) + \frac{k}{2} \sum_{j \neq i} e(V_j) \\
&\quad = \frac{k}{2} \left(k e(V_i) + \sum_{j=1}^{k} e(V_j) \right) \\
&\quad = \frac{k}{2} \bigl(k e(V_i) + m - e(V_1, \ldots, V_k) \bigr) \\
&\quad \leqslant \frac{k}{2} \left(k \left(\frac{m}{k^2} + \frac{k-1}{2k^2} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right) \right) + m - \left(\frac{k-1}{k} m + \frac{1}{2k} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right) \right) \right) \\
&\quad = m + \frac{k-2}{4} \left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2} \right). \qquad \square
\end{aligned}$$

Note that if in the proof of Corollary 4.1 we could ask

$$e(V_1,\dots,V_k) \geqslant \frac{k-1}{k}m + \frac{k-1}{2k}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right)$$

then the term $\frac{k-2}{4}(\sqrt{2m+1/4}-1/2)$ in Corollary 4.1 would vanish. So an affirmative answer to Problem 1.2 with the above inequality in place of (1.1) implies an affirmative answer to the following problem of Bollobás and Scott [6] with $c(k)=k/2$.

Problem 4.2. What is the largest $c(k)$ so that every graph G with m edges has a partition of $V(G)$ into $V_1,\dots,V_k$ such that for each $i \in \{1,\dots,k\}$,

$$\binom{k+1}{2}e(V_i) + c(k)\sum_{j\neq i} e(V_j) \leqslant m?$$

In [6], Bollobás and Scott note that $c(k)=k/2$ (if true) would be best possible. Indeed, using Theorem 1.4, we can show that for sufficiently large graphs, $c(k)$ can be arbitrarily close to $k/2$.

Corollary 4.3. *Let G be a graph with m edges, and let $k \geqslant 1$ be an integer. Then for any positive real number ϵ, there exists an integer $m(k,\epsilon)$ such that if $m \geqslant m(k,\epsilon)$ then $V(G)$ admits a partition into $V_1,\dots,V_k$ such that for each $i \in \{1,\dots,k\}$,*

$$\binom{k+1}{2}e(V_i) + \frac{k}{2+\epsilon}\sum_{j\neq i} e(V_j) \leqslant m.$$

Proof. If $k=1$, the assertion holds trivially. So we may assume that $k \geqslant 2$.

By Theorem 1.4, $V(G)$ admits a partition into $V_1,\dots,V_k$ such that

$$e(V_1,\dots,V_k) \geqslant \frac{k-1}{k}m + \frac{1}{2k}h(m),$$

and for each $i \in \{1,2,\dots,k\}$,

$$e(V_i) \leqslant \frac{m}{k^2} + \frac{k-1}{2k^2}h(m).$$

Therefore, for each $i \in \{1,2,\dots,k\}$,

$$\begin{aligned}
&\binom{k+1}{2}e(V_i) + \frac{k}{2+\epsilon}\sum_{j\neq i} e(V_j)\\
&= \frac{k}{4+2\epsilon}\left((2k+\epsilon k+\epsilon)e(V_i) + 2\sum_{j=1}^{k} e(V_j)\right)\\
&= \frac{k}{4+2\epsilon}\big((2k+\epsilon k+\epsilon)e(V_i) + 2\big(m - e(V_1,\dots,V_k)\big)\big)\\
&\leqslant \frac{k}{4+2\epsilon}\left((2k+\epsilon k+\epsilon)\left(\frac{m}{k^2}+\frac{k-1}{2k^2}h(m)\right) + 2\left(m - \frac{k-1}{k}m - \frac{1}{2k}h(m)\right)\right)\\
&= m - \left(\frac{(k-1)\epsilon}{k(4+2\epsilon)}m - \frac{(2k+\epsilon k+\epsilon)(k-1)-2k}{2k(4+2\epsilon)}h(m)\right).
\end{aligned}$$

Since $h(m)=\sqrt{2m+1/4}-1/2$, it is easy to see that there exists an integer $m(k,\epsilon)$ such that if $m \geqslant m(k,\epsilon)$, then

$$\frac{(k-1)\epsilon}{k(4+2\epsilon)}m - \frac{(2k+\epsilon k+\epsilon)(k-1)-2k}{2k(4+2\epsilon)}h(m) \geqslant 0.$$

So when $m \geqslant m(k,\epsilon)$, we have $\binom{k+1}{2}e(V_i) + \frac{k}{2+\epsilon}\sum_{j\neq i} e(V_j) \leqslant m$. $\square$

It would be nice to know whether the methods in this paper can be modified to show that the $c(k)$ in Problem 4.2 is $k/2$.

Acknowledgments

Xingxing Yu wishes to thank Brian Nakamura for his helpful discussion on the $k = 3$ case of Problem 1.3. We also thank an anonymous referee for helpful comments and pointing out additional references.

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